Adaptive Generalized Synchronization of Complex Networks

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Abstract. We propose a synchronization strategy that adaptively adjust the coupling strength of a network of strictly different dynamical systems as a way to achieve generalized synchronization (GS). There are two basic approaches to detect GS between systems: In the first, GS is inferred from the identical synchronization of two systems under the same driving force, the so-called auxiliary system approach. Alternatively, a functional relationship between the systems can be explicitly imposed by design, this is usually called the controlled synchronization approach. In this contribution, we take the latter approach to impose GS on a linearly and diffusively coupled network, where the nodes are different dynamical systems that are fully triangularizable and the coupling strength of the network is adjusted adaptively. We illustrate our results with numerical simulations.

Keywords: Complex networks, Generalized synchronization, Adaptive control.

1 Introduction

In recent years, the synchronization of complex networks has received a great deal of attention from the scientific community (Wu, 2002; Boccaletti et al., 2006). These investigations are motivated by the necessity of an improved understanding of the synchronization phenomenon in real-world complex networks, such as the Internet, the WWW, biological and social networks, among others (Arenas et al., 2008; Newman, 2010). A particularly significant concern is to study the synchronization of chaotic systems coupled in complex topologies (Wang and Chen, 2002a; Wang and Chen, 2002b). The main concert of these investigations has been the emergence of identical synchronization in networks consisting of identical $n-$dimensional dynamical systems. However, in real-world situations a far more likely scenario is that the nodes be different systems with parameter uncertainties and disturbances. In these situations, identical synchronization is unlikely. Yet, even under these situations networks exhibit some form of temporal correlation, phenomena like interdependence, auto-organization, consensus and collaboration are ever present is complex networks. Clearly these interactions go beyond simultaneous dynamical evolution as prescribed in identical synchronization, these phenomena require a more general definition of synchronization.

The concept of generalized synchronization (GS) was originally defined in the literature for the master-slave synchronization of chaotic systems (Abarbanel et al., 1996). Between two systems GS refers to the existence of a functional relationship between their dynamical states (Boccaletti et al., 2002). Different types of GS can be defined, depending how the state space of one node are mapped to the others. In this way, one can think of complete identical synchronization as a particular case of GS where the functional relationship is the identity. Another form of GS is achieved when the functional relationship is defined in terms of coordinate transformations, for example a diffeomorphism defined on a feedback linearization (Femat et al., 2005). In the literature we have basically to approaches to identify GS: An indirect method, in which synchronization in generalized terms is inferred from the identical synchronization of two systems under the same driving force, the so-called auxiliary system approach (Abarbanel et al., 1996). Alternatively, GS can be directly achieved by controllers that impose a prescribed functional relationship between the systems, this approach is usually called the controlled synchronization method. The main difference between these approaches is whether or not the description of the functional relationship between the nodes is of significance. In the case of auxiliary system approach, its existence is implied, while in controlled synchronization is a requirement. Recently the concept of GS has been extended to the case of dynamical systems coupled in complex network. In some earlier works (Hung et al., 2008; Xu et al., 2008; Liu et al., 2010) the auxiliary systems approach was considered, while others focus on the controlled synchronization approach (Guan et al., 2009).

In this contribution, we take the controlled synchronization approach to achieved GS in a network of linear and diffusively coupled exact linearizable by state feedback nonidentical n-dynamical systems. Unlike previous works on controlled GS, we propose to design an adaptive controller such that a given functional relationship between the states of different groups of nodes is imposed.

The rest of the manuscript is organized as follows. In Section II, GS problem for a network is express as a stability analysis problem. In Section III, we present our proposed adaptive controller designed for GS on a particular class of dynamical networks. In Section IV, the proposed design is illustrated with numerical simulations. Then, the contribution is closed with comments and conclusions.

2 Problem Statement

Consider a network of N non-identical nodes, with each one been a dynamical system described by

$$\dot{x}_i = f_i(x_i) \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in})^{\mathsf{T}} \in \mathbf{R}^n$ are state variables of node i (all nodes are assume to have the same dimension n); and $f_i : \mathbf{R}^n \to \mathbf{R}^n$ is a known nonlinear function describing the dynamical evolution of node i.

The state equation of the entire dynamical network is

$$\dot{x}_i = f_i(x_i) + g_i(X) + u_i \tag{2}$$

for $i = 1, 2, ..., N$ where $X = (x_1, x_2, ..., x_N) \in \mathbf{R}^{n \times N}$ are form by the state variables of all the nodes; $g_i : \mathbf{R}^{n \times N} \to \mathbf{R}^n$ are the coupling functions describing the connections to node i from the rest of the network; and $u_i \in \mathbf{R}^m$ $(m \leq n)$ is a local controller to be designed.

A dynamical network is said to be identically synchronized, if the state solutions of every node move in unison, in the sense that

$$\lim_{t \to \infty} \|x_i - x_j\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{3}$$

The synchronization criterion for complete identical synchronization, can be interpreted as requiring that the states variables of any node in the network be exactly mapped to the state variables of any other. A generalization of this interpretation of synchronization can be introduced by considering mappings between the state variables of the nodes to be different from the identity, in this way more complicated interactions between the network components can be considered (Boccaletti et al., 2002). Then, the network in (2) will be synchronized in a generalized sense with respect to the functional relation H_i if the condition

$$\lim_{t \to \infty} \|x_i - H_i(x_j)\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{4}$$

is satisfied. Note that, the functional H_i maybe the same for all the nodes or it can be different for each pair of nodes. Additionally, potentially each system can have its own transformation H_{Mi} and H_{Si}, with a GS condition

$$\lim_{t \to \infty} \|H_{Mi}(x_i) - H_{Si}(x_j)\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{5}$$

where $H_i = H_{Mi} \circ H_{Si}$.

In the sense of (Abarbanel et al., 1996), GS is achieved by the existence of H_i not by its exact description. That is, if an auxiliary system is consider to experience the same driving forces as our system and they identically synchronize to each other; then, the existence of a functional relationship can be inferred for the original system. Similarly to (Hung et al., 2008; Xu et al., 2008; Liu et al., 2010), we can extend this approach to determine GS in a dynamical network. Considering the coupling to each node in the network as an external driving, an exact replica of the network dynamics

$$\dot{\hat{x}}_i = f_i(\hat{x}_i) + g_i(\hat{X}) + u_i \tag{6}$$

for $i = 1, 2, ..., N$ where $\hat{X} = (\hat{x}_1, ..., \hat{x}_N) \in \mathbf{R}^{n \times N}$ can be taken to be an auxiliary network system.

Following the auxiliary system approach the network (2) achieves GS if (Abarbanel et al., 1996)

$$\lim_{t\to\infty} |x_i - \hat{x}_i| = 0 \tag{7}$$

for $i = 1, ..., N$ with the initial conditions $x_i(0) \neq \hat{x}_i(0)$.

From (2) and (6) the dynamics of the error $\epsilon_i = \hat{x}_i - x_i$ are given by

$$\dot{\epsilon}_i = f_i(x_i) - f_i(\hat{x}_i) + g_i(X) - g_i(\hat{X}) \tag{8}$$

for $i = 1, ..., N$. The emergence of GS is equivalent to the stability of the zero equilibrium point of (8). There are different results in the literature where GS is assured for dynamical networks under some very standard assumptions like, global Lipschitz condition for all nodes with linear and diffusive couplings; e.g. in (Liu et al., 2010) adaptive coupling strengths are used to achieve GS.

The auxiliary system approach for GS can be applied to establish that the network is synchronized in a generalized sense. However, although the functional relations H_i exist, its not possible to determine its specific form. If we want to impose a functional relationship among the nodes in the network a controlled synchronization approach is needed. We define a GS error between the i and j-th nodes as $e_{ij} = H_{Mi}(x_i) - H_{Si}(x_j)$, for $i, j = 1, ..., N$, which has the dynamics

$$\dot{e}_{ij} = H_{Mi}(f_i(x_i) + g_i(X)) - H_{Si}(f_j(x_j) + g_j(X)) + \nu_i \tag{9}$$

for $i, j = 1, ..., N$ with $\nu_i = H_{Mi}(u_i) - H_{Si}(u_j)$.

The total number of GS errors in the network can be reduced to $N - 1$ by defining $j = i + 1$, then we have

$$\dot{e}_i = H_{Mi}(f_i(x_i) + g_i(X)) - H_{Si}(f_{i+1}(x_{i+1}) + g_{i+1}(X)) + \nu_i \tag{10}$$

for $i = 1, ..., N - 1$. To stabilize (10) different approaches can be undertaken. In the following Section, we use an adaptive law to adjust the coupling strength in the network in order to achieved GS.

3 Generalized synchronization design

The design of the local controllers ν_i strongly depends on the nature of the nodes and the network topology. In this contribution we make a few simplifying assumptions.

In the first place, we will consider only nodes that either are or can be transformed into a triangular form, e.g. by a coordinate transformation and a feedback linearization controller. Although this may seem a very restrictive condition, a large number of chaotic systems can in fact be transformed to a triangular or at least partially triagularized form with internal dynamics by linearizing feedback. Additionally, chaotic dynamics can be generated from piecewise linear systems that are easily triangularizable (Sprott, 2000; Campos et al., 2010). It follows from this assumption that an adequate coordinate transform $\mathcal{T}_i$ exist for each node such that (1) can be rewritten as:

$$\dot{z}_i = A_i z_i + B\psi_i \tag{11}$$

where $z_i = \mathcal{T}_i x_i = (z_{i1}, z_{i2}, ..., z_{in})^\top \in \mathbf{R}^n$ are the transform state coordinates of node i; the constant matrices A_i and B have the controller-type companion form

$$A_i = \begin{bmatrix} 0 & 1 & 0 \dots & 0 \\ 0 & 0 & 1 \dots & 0 \\ \vdots & \vdots & \vdots \dots & \vdots \\ 0 & 0 & 0 \dots & 1 \\ a_{i,1} & a_{i,2} & a_{i,3} \dots & a_{i,n} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

with ψ_i the linearizing feedback controller, if such controller is necessary.

The second simplifying assumption is that the network topology affects the vector field of each node in a linear and diffusive way, such that,

$$g_i(Z) = \gamma_i(t) \sum_{j=1}^{N} c_{ij} \Gamma z_j \tag{13}$$

for $i = 1, 2, ..., N$ where $Z = (z_1, ..., z_N) \in \mathbf{R}^{n \times N}$ are the node states in the transformed coordinates; $\Gamma \in \mathbf{R}^{n \times n}$ is a $0-1$ inner connection matrix describing the manner in which the state variables of node i and j are connected; and $C = \{c_{ij}\} \in \mathbf{R}^{N \times N}$ is the $0-1$ coupling matrix, which captures the topological structure of the network, if $c_{ij} = 1$ $(i \neq j)$, there is a connection of strength between the nodes i and j is $\gamma_i(t)$, otherwise the nodes are disconnected. As a consequence of the diffusive coupling assumption, the diagonal entries of the coupling matrix satisfy the following equality

$$c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij} = -\sum_{j=1, j\neq i}^{N} c_{ji}, \tag{14}$$

for $i = 1, 2, ..., N$. Further, assuming that there are no isolated nodes in the network, the eigenvalues of C have a zero eigenvalue with multiplicity one, all other eigenvalues are real and negative. Under this connection structure in the transformed variables the network in (2) becomes:

$$\dot{z}_i = A_i z_i + B\psi_i + \gamma_i(t) \sum_{j=1}^{N} c_{ij} \Gamma z_j, \tag{15}$$

for $i = 1, 2, ..., N$. where ψ_i are the feedback linearizing controllers. To achieve GS in the original variables, in the transformed variables we look for complete identical synchronization of the network. In particular, since the nodes in the transformed variables have a very similar structure (11)-(12), we can argue that their differences are bounded and we can define the average node as reference for synchronization $\bar{s} = \frac{1}{N}\sum_{j=1}^{N} z_j$. The dynamics of the average node are given by

$$\begin{array}{l} \dot{\bar{s}} = \frac{1}{N}\sum_{k=1}^{N}(A_k z_k + B\psi_k) \\ \quad +\frac{1}{N}\sum_{k=1}^{N}(\gamma_k(t)\sum_{j=1}^{N} c_{kj}\Gamma z_j) \end{array} \tag{16}$$

Notice that under the assumption that the control actions vanish at the synchronized solution ($z_1 = z_2 = ... = z_N = \bar{s}$) and as a consequence of the diffusive nature of the coupling (14), once the network is synchronized the second term of (16) is also zero. Then, at the synchronized solution the dynamics of the reference node is the average of the nodes isolated from the network, in the transformed variables that is:

$$\dot{\bar{s}} = \frac{1}{N}\sum_{k=1}^{N}(A_k z_k + B\psi_k) \tag{17}$$

We define the synchronization error as $e_i = z_i - \bar{s}$, from (15) and (16) the error dynamics are given by:

$$\dot{e}_i = \mathcal{A}_i(z_i, \bar{s}) + \gamma_i(t)\sum_{j=1}^{N} c_{ij}\Gamma e_j \tag{18}$$

for $i = 1, 2, ..., N$ with

$$\mathcal{A}_i(z_i, \bar{s}) = A_i z_i + B\psi_i - \frac{1}{N}\sum_{k=1}^{N}(A_k z_k + B\psi_k)$$

Given that the term $\mathcal{A}_i$ in (18) is the difference between the current node and the reference node ($z_i - \bar{s}$). If we restrict our attention to chaotic nodes which can be traingularized. Then, is reasonable to expect that this term is bounded, that is, we assume that

$$|\mathcal{A}_i(z_i, \bar{s})| \leq \beta_i(z_i - \bar{s}) \tag{19}$$

for $i = 1, 2, ..., N$ with β_i nonnegative constants.

To stabilize the error dynamics (18) we propose to adaptively adjust the coupling strengths $\gamma_i(t)$ as described in the following result.

Theorem 1: For a network of nonidentical nodes that can be transformed into a triangularized form (11), which is linearly and diffusively coupled such that the dynamics of each node in the network are given by (15). Under the assumption (19) described above, using the following adaptive law to adjust the coupling strength of the network:

$$\dot{\gamma}_i(t) = -\alpha_i\sum_{j=1}^{N} c_{ij} e_i^{\top}\Gamma e_j \tag{20}$$

where α_i are positive constants, describing the adaptation speed, for $i = 1, 2, ..., N$. The network in the transformed variables will identically synchronized with the reference node $\bar{s}$. Equivalently, in the original variables the network will synchronizes in the generalized sense of (5), in terms of the coordinate transformations $\mathcal{T}_i$.

Proof: Using the following Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{N} e_i^\top e_i + \frac{1}{2}\sum_{i=1}^{N} \frac{1}{\alpha_i}[\gamma_i(t) + \gamma^*]^2$$

The time derivative of V along the trajectories of the error dynamics and the adaptive law is given by

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \dot{e}_i + \sum_{i=1}^{N} \frac{1}{\alpha_i}[\gamma_i(t) + \gamma^*]\dot{\gamma}_i(t)$$

From (18) and (20) we have

$$\dot{V} = \sum_{i=1}^{N} e_i^\top [\mathcal{A}_i(z_i, \bar{s}) + \gamma_i(t)\sum_{j=1}^{N} c_{ij}\Gamma e_j] - \sum_{i=1}^{N}[\gamma_i(t) + \gamma^*][\sum_{j=1}^{N} c_{ij}e_i^\top \Gamma e_j]$$

Using (18) we get

$$\dot{V} \leq \sum_{i=1}^{N} e_i^\top \beta_i e_i + \sum_{i=1}^{N} \gamma_i(t)e_i^\top \sum_{j=1}^{N} c_{ij}\Gamma e_j - \sum_{i=1}^{N} \gamma_i(t)e_i^\top \sum_{j=1}^{N} c_{ij}\Gamma e_j - \sum_{i=1}^{N} \gamma^* \sum_{j=1}^{N} c_{ij}e_i^\top \Gamma e_j$$

Letting $\beta = \max\{\beta_i | i = 1, 2, ..., N\}$ and $k_i = \sum_{i=1, j\neq i}^{N} c_{ij}$ be the largest bound and the node degree, respectively. Additionally, considering that C is a symmetric matrix, we can maneuver the indexes in the last term to get

$$\dot{V} \leq \beta\sum_{i=1}^{N} e_i^\top e_i - \gamma^* \sum_{i=1}^{N} k_i e_i^\top \Gamma e_i \tag{21}$$

Defining the error vector $E = [e_1^\top, e_2^\top, ..., e_N^\top]^\top$ and the matrix $P = K \bigotimes \Gamma$, with $K = \mathrm{Diag}(k_i, k_2, ..., k_N)$ and $\bigotimes$ the Kronecker product. The inequality in (21) can be rewritten as:

$$\dot{V} \leq \beta E^\top E - \gamma^* E^\top P E \leq E^\top[\beta - \gamma^* \lambda_{min}(\frac{P + P^\top}{2})]E$$

It follows that by letting γ^* be a sufficiently large positive constant, we have that

$$\dot{V} \leq E^\top E$$

Then, the error dynamics in (18) are asymptotically stable about the zero fixed point ($e_i = 0$) when the coupling strengths are adjusted according to (20), which implies that the network in the transformed variables achieves identical complete synchronization. In consequence, the dynamical network in the original coordinates achieves GS in the sense of (5), with respect to the coordinate transformations $\mathcal{T}_i$.

Q.E.D.

4 Illustrative example

We consider a network with two different types of nodes which can be triangularized. Namely, Sprott circuits (◯)(Sprott, 2000):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -0.6x_3 - x_2 - 1.2x_1 + 2sgn(x_1) + u \end{aligned} \tag{22}$$

which is already in triangular form. The other type of nodes are Rössler systems (□) :

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + 0.1x_2 \\ \dot{x}_3 &= x_3(x_1 - 14) + 0.1 + u \end{aligned} \tag{23}$$

The different nodes are connected randomly according to the ER network model

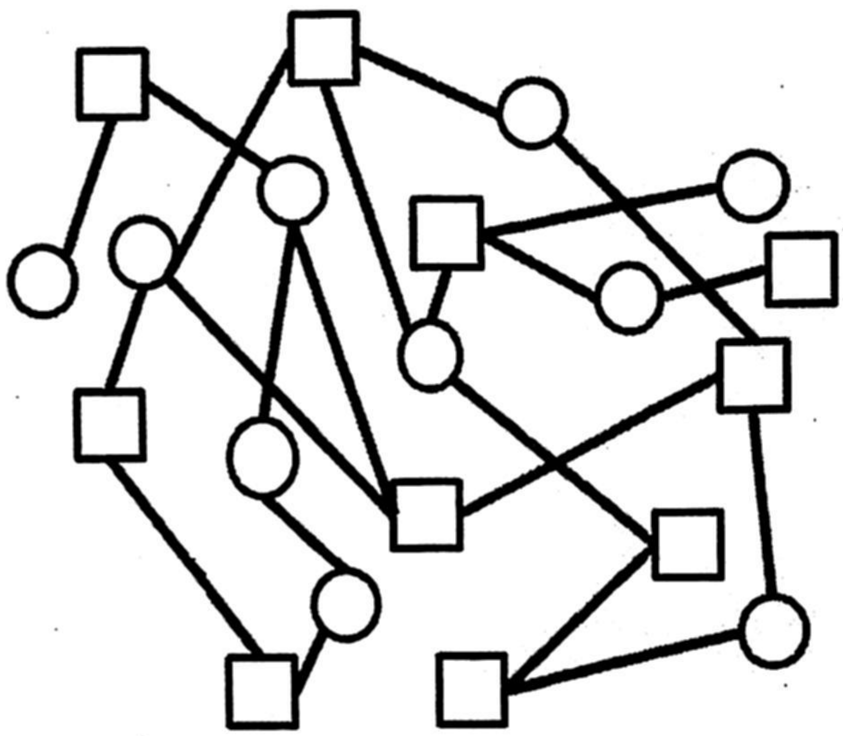

Fig. 1. Network of non-identical nodes (Sprotts:◯ and Rössler:□).

(Newman, 2010). A possible realization with twenty nodes is shown in Figure 1. To achieve GS in the network we use a coordinate transformation for the Rössler system. Assuming that the output of (23) is $y = x_2$, the following coordinate transformation takes the Rössler system to a triangular form with the transform variables $z = \phi(x)$ (Femat et al., 2005):

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + 0.1x_2 \\ 0.1x_1 + (0.1^2 - 1)x_2 - x_3 \end{pmatrix} \tag{24}$$

This coordinate transformation is a diffemorphism and its inverse is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_2 - 0.1z_1 \\ z_1 \\ 0.1z_2 - z_3 - z_1 \end{pmatrix} \tag{25}$$

To achieve GS in the network coupling strengths are adjusted adaptively according to Theorem 1. In Figure 2 the trajectories of the Sprott circuits in the the Rössler in its transformed coordinates are shown. In Figure 3, the synchronization error in the original coordinates is presented, as shown the adaptive coupling strength produces an identical synchronization of the network in the transformed variables. As such, GS with the mapping function H_i is obtained on the network. The error in the original coordinates is shown in Figure 3.

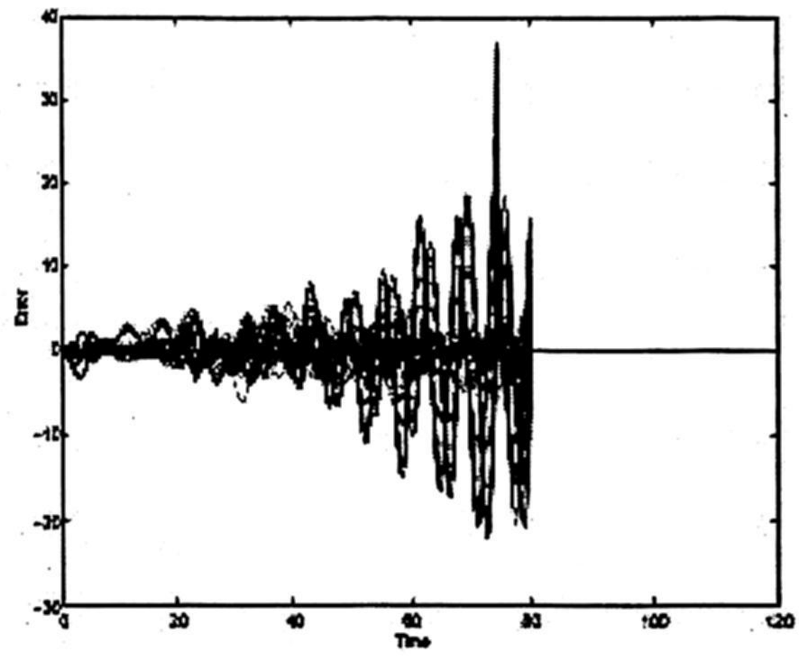

Fig. 2. Synchronization error on the transformed coordinates $z(t)$

5 Conclusions

On networks with different dynamical nodes complete identical synchronization is not directly achievable. As such, alternative interpretations of the synchronization phenomena are necessary. In this contribution, we investigate the emergence of GS in a network of non identical nodes via adaptive control. In particular, we consider nodes that can be exactly linearized by state feedback. Under such conditions, GS can be achieved by imposing a functional relationship between the nodes in the network. There are limitations of the proposed method, for example the necessity of triangularizing state feedback controllers. Additionally, the imposed functional relationship between the nodes is fixed by coordinate

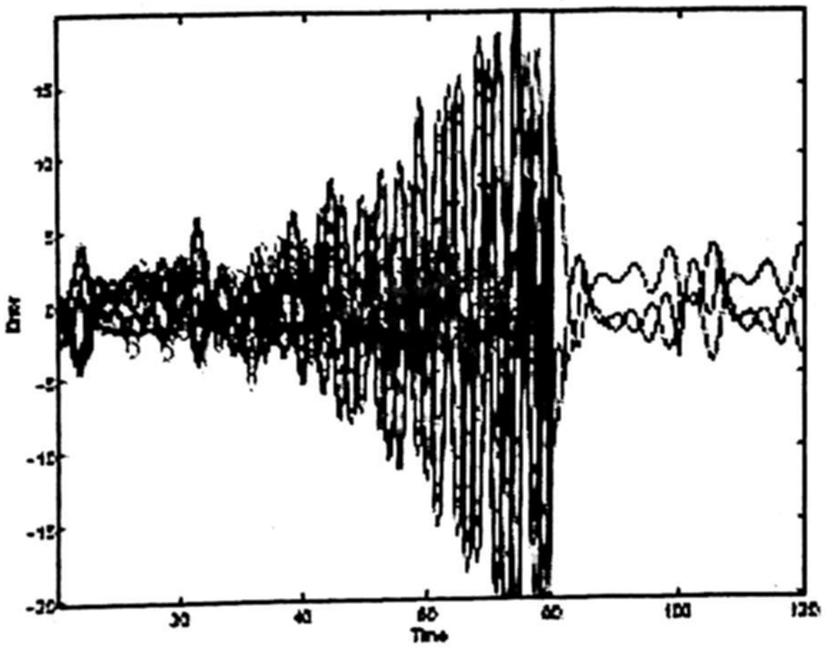

Fig. 3. Synchronization error on the original coordinates $x(t)$

transformation. However, it seems possible to overcome these restrictions by considering alternative ways to design the synchronizing controllers. These are considerations for future work and will be reported elsewhere.

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